MOTION ON A LATTICE

By C. Hewitt Dix

Department of Mathematics, The Rice Institute,

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I. *Introduction.*—Courant, Friedricks and Lewy[1] have outlined a procedure for solving boundary value problems of partial differential equations by considering the corresponding partial difference equations and allowing the mesh width to approach zero at some convenient stage of the game. The purpose of this paper is to take over some of their definitions and concepts and construct a lattice theory with no thought of any process in which the mesh width continually approaches zero.

II. 1. *A Calculus of Variations Problem.*—A necessary condition that the function $u(x_i)$ must satisfy when[2]

$$J \equiv \sum_{i=1}^{n} F(x_i, u(x_i), u_x(x_i)) \Delta x_i \tag{1}$$

is stationary is,

$$\frac{\partial F(x_i, u(x_i), u_x(x_i))}{\partial u} - \left[\frac{\partial F(x_i, u(x_i), u_x(x_i))}{\partial u_x}\right]_{\bar{x}} = 0. \tag{2}$$

This result is obtained very simply by a process everywhere similar to the process for obtaining Euler's equations from the condition

$$\delta \int_a^b F(x, y(x), y'(x))dx = 0, \tag{3}$$

excepting that instead of an ordinary integration by parts, an Abel's rearrangement is made of the terms in the finite sum.

If we use the obvious extension of (2) to more independent variables, then the expressions for $L(u)$ and $M(v)$ given by C. F. L.[3] come immediately from the hypothesis

$$\delta \sum\sum_{(G_L)} B(u, v) \Delta x_i \Delta y_i = 0. \tag{4}$$

2. *Mechanics.*—The similarity between the forms of (2) and of the Euler-Lagrange equations of mechanics suggests an investigation of a dynamics based on (2). There is also the problem of wandering motion on a lattice in which (2) is involved directly and so one may hope for a probability interpretation of mechanics via these processes—a desirable result from the point of view of the quantum physicist.

For the case of the analogs of Laplace's equation and the wave equations the theory has been developed by C. F. L. The solution for the wave equation for finite differences has been given by Boole[4] and more generally by G. C. Evans.[5] This solution is identical with the solution of the differential wave equation—a striking result. Any wave differential equation may be replaced by the corresponding difference equation. The difference between the two cases will appear only when one can observe the difference between very small finite differences and differentials, i.e., no observation can choose one formulation and reject the other.

2.1 *Wandering Motion in a Linear Lattice.*—Consider a linear lattice of points whose coördinates can be specified by . . . x_{-2}, x_{-1}, x_0, x_1, x_2, . . . Let the lattice distance be 1. At each point let there be a clock and let all of the clocks be synchronous. Suppose there are particles at some of the lattice points. At the end of every unit of time each particle moves to a neighboring lattice point. A given particle is just as likely to pass to one neighbor as to another. Two particles can never occupy the same lattice point at the same time. Alternate lattice points are marked "+" and "−." The "+" lattice has particles that are completely independent of the particles in the "−" lattice. Hence, we may suppose that initially only the "+" lattice has particles. Let $u(x_i, t_i)$ be the probability that a particle will be at x_i at time t_i. Then

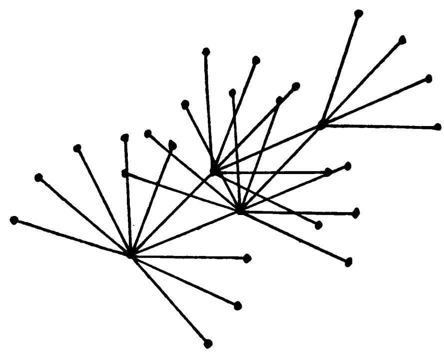

FIGURE 1
An n-way lattice.

$$u(x_i,\ t_j) = \frac{u(x_{i-1}, t_{j-1}) + u(x_{i+1}, t_{j-1})}{2}. \qquad (5)$$

Using the difference quotient notation, (5) may be written in the form

$$u_{x\bar{x}}(x_i,t_j) - u_{t\bar{t}}(x_i,t_j) - u_{\bar{t}}(x_i,t_j) - u_t(x_i,t_j) = 0 \qquad (6)$$

supposing $\Delta_x = \Delta_t = 1$. It may be observed that (6) is the analog of the telegraphist's equation. A solution of (5) or (6) is obviously $u = 1$ for "+" points and $u = 0$ for "−" points.

2.2 *Wandering Motion in n Directions.*—C. F. L.[6] consider the problem of wandering motion on a square lattice. In such a case a wandering particle can move in four possible directions. The methods still apply when the lattice is more general and the number of directions is any number.

Consider a point P and about P draw a circle of radius h. Divide the circumference into n equal parts and draw radii from P to the centers of those parts. At the end of each such radius repeat the process so that n sets of n parallel radii are formed. Repeat this process without end, thus forming an n-way lattice (see Fig. 1). If n is odd there is no negative of any direction. If n is even we may still suppose that every possible direction is positive. Then every lattice point is determined by n coordinates $(a_1, a_2, \ldots, a_n) \equiv [\alpha]$. The probability that a wandering particle will be at $[\alpha]$ at the time $t+1$ is $E_{t+1}[\alpha]$. Then $E_{t+1}[\alpha]$ is the average value of the probability that the wandering particle will be at the neighbors of $[\alpha]$ at the time t; i.e.,

$$E_{t+1}[\alpha] = \frac{1}{n}\sum_{i=1}^{n} E_t[\alpha_i] \tag{8}$$

where $[\alpha_i] \equiv (a_1, a_2, \ldots, a_i + h, \ldots, a_n)$. Subtract $E_t[\alpha]$ from both sides of (8).

$$E_{t+1}[\alpha] - E_t[\alpha] = \frac{1}{n}\sum_{i=1}^{n} (E_t[\alpha_i] - E_t[\alpha]). \tag{9}$$

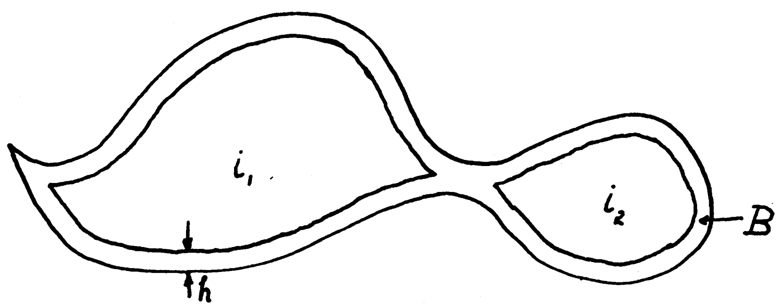

FIGURE 2

Interior sets (i_1 and i_2) and boundary (B).

If n is even, (9) reduces to

$$n(E_{t+1}[\alpha] - E_t[\alpha]) = \sum_{j=1}^{n/2} (E_t[\alpha])_{r_j\bar{r}_j}. \tag{10}$$

The expectation that a wandering particle will come to $[\alpha]$ is

$$v[\alpha] = \sum_{t=0}^{\infty} E_t[\alpha]. \tag{11}$$

So if we sum (10) from $t = 0$ to ∞ we obtain

$$-nE_0[\alpha] = \sum_{j=1}^{n/2} (v[\alpha])_{r_j\bar{r}_j}. \tag{12}$$

$E_0[\alpha] = 0$ if there was no particle at $[\alpha]$ initially, otherwise $E_0[\alpha] = 1$. The solution is then the Green's function for (12).[7]

2.3 *The Limiting Case for* $n \longrightarrow \infty$.—Equation (12) may be written in the form

$$v[\alpha] = \frac{1}{n}\sum_{i=1}^{n} v[\alpha_i] + E_0[\alpha]. \tag{13}$$

When $n \longrightarrow \infty$, (13) $\longrightarrow$ (14).

$$v[\alpha] = \frac{1}{2\pi}\int_0^{2\pi} v[\alpha']d\Theta + E_0[\alpha]. \tag{14}$$

It is interesting to notice that (14) has a form similar to an equation satisfied by the principal solution of Poisson's equation for an arbitrary distribution of finite positive mass given by G. C. Evans:[8]

$$U(P) = \frac{1}{2\pi}\int_0^{2\pi} U(M)d\Theta + \lim_{\rho_2 = 0}\,(1/2\pi)\int_{\rho_2}^{\rho_1} (1/\rho)\Phi(C_\rho)d\rho. \tag{15}$$

However, $U(P)$ and $v[\alpha]$ have quite different meanings, the mean value in the one case being taken around any circle of small enough radius to remain within the region, whereas the mean value in the other case is taken about a circle of radius h only.

In the case of equation (13) it is interesting to note the full significance of a boundary value problem. Extending the definition of the set of boundary points given by C. F. L.[9] we find the boundary is a strip whose least width is h (see Fig. 2).

2.4 *The General Lattice in 3-Space.*—If we try to extend the above work to 3-space we find that we must restrict ourselves to the directions that are perpendicular to the centers of the faces of the regular solids. If we try to use more directions we find we cannot weight the directions equally. Very likely such a study will furnish methods for dealing with crystal problems.

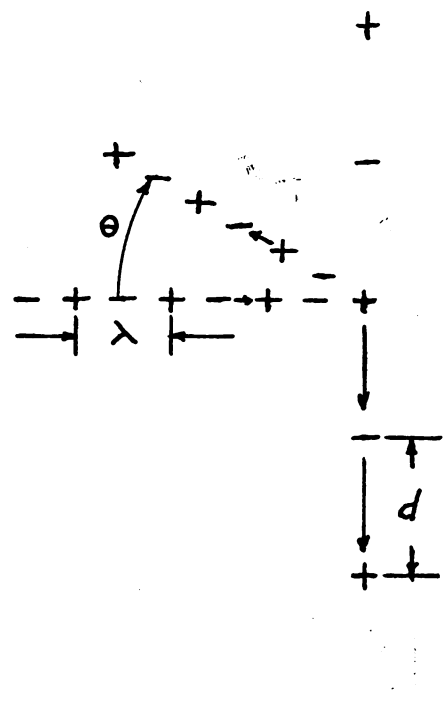

FIGURE 3

3. *Application to a Diffraction Problem.*—Imagine a cubical crystal lattice with lattice distance d. Mark neighboring points with opposite signs (+ or −). Suppose that initially all of the particles have "+"

positions. Let these particles wander in the manner of §2.1. The x-ray is also imagined as a wandering particle, but it is wandering down a linear lattice with lattice distance $\lambda/2$. The wandering particle on the x-ray lattice wanders into the lattice of the crystal. Then we assume conservation of momentum for the wandering x-ray particle plus the wandering crystal particles in a direction perpendicular to the linear x-ray lattice. Each wandering crystal particle has a momentum which is

$$\frac{\Delta \text{ action}}{\Delta \text{ distance}} = \frac{h}{2d}.$$

If it happens the number of crystal particles moving in one sense is n more than the number moving in the opposite sense then the total momentum in the given direction is $nh/2d$. The momentum of the x-ray particle is h/λ in the direction of its motion and $(h/\lambda) \sin \Theta$ in the sense opposite to the sense of the motion of the excess of crystal particles. Conservation of momentum then implies

$$nh/2d = (h/\lambda) \sin \Theta \text{ or } n\lambda = 2d \sin \Theta. \tag{16}$$

The above treatment is somewhat similar to that given by Duane and Compton.[10] The "wandering" concept is, however, new, if I am not mistaken. It seems desirable to have as many mechanisms as possible in mind for such a critical phenomenon.

[1] Courant, Friedrichs and Lewy, *Math. Annalen*, **100**, 32–74 (1928). This paper will be referred to hereafter as C. F. L.

[2] The definitions of u_x and $u_{\bar{x}}$ are given by $u_x(x) = (u(x + h) - u(x))/h$, $_{\bar{x}}(x)u = (u(x) - u(x - h))/h$. See C. F. L., page 34.

[3] C. F. L. See page 35.

[4] G. Boole, *A Treatise on the Calculus of Finite Differences*, London, **1860**, page 186.

[5] G. C. Evans, *Cambridge Colloquium Am. Math. Soc.* See pages 99–100.

[6] C. F. L. See pages 42–47.

[7] C. F. L. See pages 40–41.

[8] G. C. Evans, *Am. J. Math.*, **51**, 5 (1929).

[9] C. F. L. See page 34.

[10] W. Duane, *Proc. Nat. Acad. Sci.*, 9 (May, 1923); and A. H. Compton, *Ibid.*, (November, 1923). See also a paper (*Ibid.*, (July, 1923)) by G. Breit. It is interesting to note that Breit gives what might be termed the wave analog of the present method.